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Basic lines, axes and geometric modeling on implicit algebraic surfaces

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Abstract

We introduce some new concepts such as basic lines and divide the blending problems into two classes, Class I blending and Class II blending. We further develop the results [Li, A software system on blending of pipe surfaces, Master Thesis, Jilin University, June, 2000; Wu et al., Blending of implicit algebraic surfaces, Proceedings of the ASCM 1995, Beijing, China, August 18–20, 1995, pp. 125–131; Wu and Han, Newton form formulae for the n-way blending of quadratic surfaces, Proceedings of IJCC Workshop on Digital Engineering, August 21–22, 2003; Wu and Zhou, On blending of several quadratic algebraic surfaces, Comput. Aided Geom. Design 17(9) (2000) 759–766; Wu, Zhou and Feng, Blending two quadratic algebraic surfaces with cubic surfaces, Proceedings of ASCM, 1996, Kobe, Japan, August 20–22, 1996, pp. 73–79] and analyze the geometric condition for the Class I blending problem by using characteristic roots. We obtain the lowest degree of Class II blending according to the different position of the axes of the primary surfaces. And we derive the efficient parameterization method for a kind of cubic blending surfaces on the basis of the results obtained by Berry and Patterson [Implicitization and parameterization of nonsingular cubic surfaces, Comput. Aided Geom. Design 19 (2001) 723–738].

Keywords: Basic lines; Axes; Blending; Parameterization

1. Introduction

One of the important problems in CAGD is the geometric design based on the blending surfaces. As we know, there are two kinds of useful surfaces in CAGD, parametric surfaces (say, NURBS) and implicit

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algebraic surfaces. Compared with the parametric surfaces, the implicit algebraic surfaces can grasp all elements in the class of algebraic surfaces, they are the closure under several geometric operations and they have important applications in pipe design and corner smoothing. Because the implicit algebraic surfaces have so many advantages, the deep study on the blending of implicit algebraic surfaces is needed.

During the last 15 years, the blending of implicit algebraic surfaces attracts more and more attention. A lot of pure algebraic tools, such as ideal theory [4], Wu's Method [9] and Groebner bases method [5–8], have been used to deal with the blending problems. However, the above algebraic methods always lead to complicated nonlinear polynomial systems that are hard to solve. The complexity of the computation for the blending problem makes the method impractical for modeling designers.

On the other hand, NURBS are regarded as the milestone in the history of geometric design, which offer one common mathematical form for both standard analytical shapes and free form shapes. Due to the flexibility and effectiveness of NURBS, they became ISO industry standard tools for the representation and design of geometry in 1990. Many works [1,3] discuss the translation (implicitization and parameterization) between implicit algebraic surfaces and rational surfaces. By now it is open and difficult problem for geometric designers to generate implicit defined algebraic surfaces in an effective way such as the rational parameterization.

The main feature of CAGD is geometry. It is a very natural idea to analyze the geometric feature of implicit algebraic surfaces and then discuss the relationship between the geometric feature and the blending and the corresponding parameterization.

In this paper we will discuss two kinds of blending problems according to the different freedom of the clipping planes. The first kind of blending is Class I blending, which is to construct the blending surfaces of the lowest degree when the primary surfaces and the clipping planes are given. And the second kind of blending is Class II blending, which is to seek the blending surfaces of the lowest degree and the corresponding clipping planes when the primary surfaces are given. Obviously, the Class II is more interesting and useful in practice than Class I because most designers are not interested in the intersection decided by the clipping planes but in the blending surfaces themselves. At the same time, it is harder to solve Class II problems than Class I due to its more variables. For example, the cubic G^1 blending surfaces $S(f)$ can be written in the form

$$f = u_1 g_1 + a_1 h_1^2 = u_2 g_2 + a_2 h_2^2,$$

where u_i, a_i are linear polynomials. Both of the two kinds of blending problems will lead to complicated symbolic computation. The above nonlinear system to the Class I blending problem includes 16 variables, 20 linear equations and 28 parameters, and the same system to the Class II blending problem includes 24 variables, 20 linear equations and 20 parameters. Our purpose is to seek the condition for the existence of solutions of the above system and to obtain some kind of solutions. Most of the papers on the blending devoted to the Class I blending by using algebraic methods. We will discuss the two classes of problems by a geometric method. And then, we develop the parameterization method in [1] and obtain an efficient parameterization method and apply the method to our blending problems.

For the convenience of application, we always consider the problem in $R[x, y, z]$. Let $S(f)$ denote the algebraic surface determined by the polynomial equation $f(x, y, z) = 0$. Assume that F is a set of polynomials, and denote by $S(F)$ the set of solutions of the system of all polynomials in F. Let $\langle g, h\rangle$ be the ideal generated by the polynomials g and h.

All of our problems will be discussed under the following assumptions. Let $S(g_i)$ be primary irreducible quadratic surfaces, $S(h_i)$ be different planes and $S(g_i, h_i)$ be irreducible planar quadratic curves. Denote

$I_{ij} = \langle h_i, h_j \rangle$, $I = \langle h_1, h_2 \rangle$ and $(f)_I$ the canonical polynomial of the polynomial f with respect to the ideal I. The canonical polynomial is unique for any $f \in R[x, y, z]$. We can calculate the canonical polynomial by the division algorithm in $K[x_1, \ldots, x_n]$.

Some new concepts will be introduced before the formal discussion:

Basic line: we call $S(I_{ij}) = S(h_i, h_j)$ the basic line if the plane $S(h_i)$ intersects with plane $S(h_j)$.

Characteristic roots: we name the roots of the quadratic canonical polynomial of one variable by characteristic roots of g_i.

2. Class I blending

Using the basic line and characteristic roots, we discuss the two-way Class I blending problem and obtain some new geometric conditions which are different from the algebraic conditions we obtained in [5,7,8]:

Proposition 1. *There exists an irreducible quadratic surface $S(g)$ containing $S(g_i, h_i)(i = 1, 2)$ if and only if the characteristic roots of g_1 are the same as these of g_2.*

Proof. Just as stated in [5], there exists an irreducible quadratic surface $S(g)$ containing $S(g_i, h_i)$ if and only if there exist a nonzero real number l and linear functions a_1, a_2 such that

$$g = g_1 + a_1 h_1 = l g_2 + a_2 h_2.$$

We compute the canonical polynomial of the polynomials in the two sides of the above equation with respect to the ideal I and obtain $(g_1)_I = l(g_2)_I$, namely, the characteristic roots of g_1 are the same as g_2.

Conversely, if the characteristic roots of g_1 are the same as g_2, then there exists a nonzero real number l such that $(g_1)_I = l(g_2)_I$ which tells us $g_1 - l g_2 \in I$. Because I is prime, we obtain the following $g_1 + a_1 h_1 = l g_2 + a_2 h_2$ with linear function a_i by using the properties of Groebner bases which completes the proof of Proposition 1. $\square$

Proposition 2. *Suppose that there do not exist quadratic surfaces $S(g)$ containing $S(g_i, h_i)(i = 1, 2)$ and $\deg(g_i)_I = 2$. If there exists a cubic G^1 blending surface $S(f)$, then one of the characteristic roots of g_1 is the same as those of g_2, and the other root is different.*

Proof. Let the characteristic roots of g_i be $S(g_i, h_1, h_2) = \{r_{i1}, r_{i2}\}(i = 1, 2)$. As shown in [5], the cubic G^1 blending surface $S(f)$ must be written in the form

$$f = u_1 g_1 + a_1 h_1^2 = u_2 g_2 + a_2 h_2^2,$$

where u_i, a_i are linear functions. Computing the canonical polynomial of the above equation with respect to the ideal I, we can get

$$(u_1)_I (g_1)_I = (u_2)_I (g_2)_I.$$

For the sake of convenience, let z be the variable of $(f)_I$ and real numbers z_i be the roots of the linear equations $(u_i)_I = c_i(z - z_i) = 0(i = 1, 2)$. Applying Proposition 1 to the problem, we see

$(g_i)_I = c_i(z - z_i)(z - z_0)$. In other words, the following relationship between the two groups of characteristic roots of g_i holds: $r_{11} = r_{21} = z_0$, $z_1 = r_{12} \neq r_{22} = z_2$. □

Proposition 3. *Suppose $S(g_1)$, $S(g_2)$ are primary pipe surfaces (including cylinders, spheres, circular cones, circular paraboloid and hyperboloid of one or two sheets), the axes of $S(g_i)$ are perpendicular to the clipping planes $S(h_i)$, respectively. If the axes of $S(g_i)$ are skew and the intersection of the two intersection curves $S(g_i, h_i)$ is empty, then there do not exist quadratic surfaces $S(g)$ containing $S(g_i, h_i)(i = 1, 2)$ and cubic G^1 blending surfaces $S(f)$.*

Proof. On the basis of the results in [2], there do not exist quadratic surfaces $S(g)$ containing $S(g_i, h_i)$ if the axes of $S(g_i)$ are skew. Suppose we find some cubic G^1 blending surfaces $S(f)$, then we can get a real number z_0 such that $(g_i)_I(z_0) = 0$, the corresponding point P_0 on the basic line such that $P_0 \in S(g_1, h_1, h_2) \cap S(g_2, h_1, h_2)$, which brings a conflict with the fact that $S(g_1, h_1) \cap S(g_2, h_2) = \emptyset$. So the hypothesis is wrong and the proposition holds. □

3. Class II blending

Then we turn to the Class II blending problem.

Let $S(g_i)$ be primary pipe surfaces (including cylinders, circular cones, circular paraboloid and hyperboloid of one or two sheets), the perpendicular intersection of the primary pipe surface and the corresponding clipping plane be circle. Denote L_i the axis of $S(g_i)$. We apply the results obtained in [6] to the Class II blending problem and derive the following theorems.

Theorem 1 (*Two-way Class II blending*). *If the axes L_1 and L_2 intersect or equal, we can construct cubic G_1 blending surfaces; otherwise, the axes L_1 and L_2 are parallel or skew, we can obtain quadratic G^1 blending surfaces.*

Proof. Warren [4] constructs two families of quartic blending surfaces: $f = g_1 g_2 + l h_1^2 h_2^2$ or $f = g_1 h_2^2 + m g_2 h_1^2$, where l and m are positive constant numbers. To finish the proof of the Theorem, we should construct cubic blending surfaces to the first two cases. Wu and Han [6] discussed the Class I blending problem and derived that if $\text{Condition}_1 = \text{Condition}_2 = r^2$, then there exist cubic G^1 two-way blending surfaces, where

$$\text{Condition}_i = \begin{cases} r_i^2 + d_i^2 \ \text{(cylinder)}, \\ d_i^2 + q_i^2(d_i - x_{0i})^2 \ \text{(circular cone)}, \\ d_i^2 + 2p_i\,(d_i - x_{0i}) \ \text{(circular paraboloid)}, \\ d_i^2 + l_i^2 + q_i^2(d_i - x_{0i})^2 \ \text{(hyperboloid of one sheet)}, \\ d_i^2 - l_i^2 + q_i^2(d_i - x_{0i})^2 \ \text{(hyperboloid of two sheets)}. \end{cases}$$

When we study the Class II blending problem, the key point is to choose the clipping planes satisfying the condition for the existence of the cubic G^1 blending surfaces. To our problem, the condition is equivalent to solving two quadratic equations in d_i. Detailed analysis shows that we always find suitable d_i satisfying the above condition when r^2 is sufficiently large.

Theorem 2 (*n-way Class II blending*). *If the intersection of* $L_i\,(i=1,\ldots,n)$ *is a point, then there exist* G^1 *n-way blending surfaces of degree* $n+1$ *defined by*

$$f = ug - h_1h_2\cdots h_n,$$
$$u = \sum_{i=1}^{n} b_i\, h_1\cdots h_{i-1}h_{i+1}\cdots h_n,$$
$$g = x^2+y^2+z^2-r^2,$$

where $h_i = x^{(i)} - d_i$, $x^{(i)}$*-axis is just the axis* $L_i\,(i=1,\ldots,n)$, d_i *and* b_i *are relevant to the geometric parameters of the primary surface, which can be computed directly.*

Proof. The proof is very similar to the proof of the Theorem 1 and enthusiastic readers can substitute $b_i = 1/2d_i$ (cylinder), $1/(2d_i + 2q_i^2(d_i - x_{0i}))$ (cone or hyperboloid) or $1/2(d_i + p_i)$ (circular paraboloid) into the form of f to test the conclusion of this theorem.

As the direct corollary of Theorem 2, we discuss the blending of $2n$ cylinders of symmetric form (i.e. the $2n$ cylinders are composed of n couples of two cylinders of the same equation) and find an interesting facts: the corresponding Class II blending surfaces degenerate to degree $2n$ instead of the usual $2n+1$ because of the special position of the primary cylinders. □

4. Parameterization for two-way cubic blending surfaces

At last, we discuss the parameterization of two-way cubic blending surfaces. Using the basic line theory, we derived a method of parameterization of blending surfaces in [6]. As the direct results of [6], we obtain that the two-way blending surface $S(f)$ containing the basic line $S(h_1, h_2)$. The shortness of the method is hard to directly get the rational parameterization of a cubic blending surface.

Because NURBS are the basic surfaces for CAD system, we have to seek an efficient way to construct the rational parameterization of cubic blending surfaces. Berry and Patterson [1] gave some good ideas for rational parameterization. They divided the whole process into four steps: the first is to find one straight line on the cubic surface, then find some other straight lines on the surface after we get the first straight line, the third is to construct a 3×3 matrix U and then another matrix called Hilbert–Burch Matrix which is 3×4 by using some ideal theory, and the last step is to obtain the parameterization of cubic surfaces from the 3×3 Hilbert–Burch Matrix.

One of the difficulties is to find some lines on cubic surfaces. This is not necessarily easy. A rough way to find one straight line on a cubic surface always means solving a system of cubic polynomials of four variables, which brings us complicated and almost impractical symbol computation. What we are interested in is the parameterization for the following cubic blending surfaces:

$$f = (b_1h_2 + b_2h_1)(x^2+y^2+z^2-r^2) - h_1h_2.$$

On the basis of the results obtained in [2], we find that the basic line $S(h_1, h_2)$ is just one line on the surface by which we successfully complete the first step of parameterization for two-way blending surfaces.

Then we need to find the other lines by using the basic line. Rotate a plane about the basic line and intersect it with the cubic surface. This is done by taking $h_1 = th_2$ in $f = 0$ and cancelling the factor h_2.

The other factor $Q(t)=0$, is quadratic in two variables in $R[x, y, z]/\langle h_1, h_2\rangle$. Without loss of generality, let $h_1 = x - d_1$, $h_2 = y - d_2$. The corresponding $Q(t)$ is of the form

$$
\begin{aligned}
Q(t) = {} & (b_2t + b_1t^2 + b_2t^3 + b_1)y^2 + (b_2t + b_1)z^2 + (-2d_2b_2t^3 + 2b_1d_1t - 2d_2b_1t^2 \\
& + 2b_2d_1t^2 - t)y + d_2^2b_2t^3 + d_2^2b_1t^2 - b_1r^2 - 2d_2b_2d_1t^2 + b_1d_1^2 + d_2t - 2d_2b_1d_1t \\
& + b_2d_1^2t - b_2r^2t.
\end{aligned}
$$

To find some other lines on the surfaces, we seek values of t for which $Q(t)$ factors into two lines in the plane $h_1 = th_2$. The possibility of the factorization is equivalent to finding the roots of the determinant of the Hessian of $Q(t)$. In general, the factorization of a polynomial $Q(t)$ is impossible by a fixed formula because it is always quintic (degree five). In our discussion, the determinant is

$$
\begin{aligned}
D(t) = {} & (2b_1 + 2b_2t)((4b_2^2d_2^2 - 4b_2^2r^2)t^4 + (4b_2d_1 - 8b_1b_2r^2 + 8b_1b_2d_2^2 - 8b_2^2d_2d_1)t^3 \\
& + (4b_1^2d_2^2 + 4b_1d_1 - 4b_2^2r^2 + 4b_2^2d_1^2 + 4b_2d_2 - 1 - 16b_1b_2d_1d_2 - 4b_1^2r^2)t^2 \\
& + (8b_1b_2d_1^2 - 8b_1b_2r^2 - 8b_1^2d_1d_2 + 4b_1d_2)t + 4b_1^2d_1^2 - 4b_1^2r^2)
\end{aligned}
$$

Obviously, $t = -b_1/b_2$ is one root of $D(t) = 0$. Thus the other part of $D(t)$ degenerates into quartic, which is possible to solve by some fixed formula. If we get the four roots t_1, t_2, t_3 and t_4 of $D(t) = 0$, we choose two of them, t_1 and t_2, which are real or conjugate complex numbers. In each of them we have $Q(t_1) = m_1m_2$ and $Q(t_2) = n_1n_2$ where m_1, m_2, n_1 and n_2 are linear function in y and z. Using the method in [1], we choose the parameters p_i and k_i in

$$
U = \begin{pmatrix} x - t_1(y - d_1) - d_2 & 0 & m_1 \\ 0 & x - t_2(y - d_2) - d_1 & n_1 \\ k_1m_1 & k_2m_2 & p_1x + p_2y + p_3z + p_4 \end{pmatrix}
$$

such that $\det(U) = f$. After getting the 3×4 matrix H from the equation

$$
U \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = H \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix},
$$

we will obtain a parameterization for the cubic surfaces

$$
x = \frac{\det(H_1)}{\det(-H_4)}, \quad y = \frac{\det(-H_2)}{\det(-H_4)}, \quad z = \frac{\det(H_3)}{\det(-H_4)},
$$

where H_i are 3×3 submatrices of H by cancelling the ith column.

For example, taking $b_1 = \frac{1}{2}$, $b_2 = \frac{1}{2}$, $d_1 = \frac{1}{2}$, $d_2 = \frac{1}{2}$, $r^2 = \frac{3}{8}$, the quintic function

$$
Q(t) = (1 + t)\left(-\tfrac{1}{8}t^4 + \tfrac{1}{4}t^3 - \tfrac{1}{4}t^2 + \tfrac{1}{4}t - \tfrac{1}{8}\right)
$$

has five roots: -1, 1, 1, i and $-i$. We choose $t_1 = i$, $t_2 = -i$ and obtain the complex matrix U as follows

$$U = \begin{pmatrix} x - \frac{1}{2} - i(y - \frac{1}{2}) & 0 & (\frac{1}{2} + \frac{1}{2}i)(z - \frac{\sqrt{2}}{4}i) \\ 0 & x - \frac{1}{2} + i(y - \frac{1}{2}) & (\frac{1}{2} - \frac{1}{2}i)(z + \frac{\sqrt{2}}{4}i) \\ (a+bi)(z + \frac{\sqrt{2}}{4}i) & (a - bi)(z + \frac{\sqrt{2}}{4}i) & p_1 x + p_2 y + p_3 z + p_4 \end{pmatrix}.$$

To get a real parameterization, we have to change the U to a real matrix

$$U^* = \begin{pmatrix} 2x - 1 & -y + \frac{1}{2} & z + \frac{\sqrt{2}}{4} \\ -y + \frac{1}{2} & -\frac{1}{2}x + \frac{1}{4} & \frac{1}{2}z - \frac{\sqrt{2}}{8} \\ -\frac{\sqrt{2}}{4} & \frac{1}{2}z & \frac{1}{2}(x + y) \end{pmatrix}$$

such that $\det(U^*) = -f$. After getting the 3×4 matrix H, the parameterization of f is

$$x = -\frac{\frac{1}{8}Y_2^3 - \frac{\sqrt{2}}{16}Y_2^2Y_3 - \frac{\sqrt{2}}{4}Y_1^2Y_3 + \frac{1}{2}Y_1^2Y_2 + \frac{1}{2}Y_1Y_3^2 - \frac{\sqrt{2}}{8}Y_3^3}{-Y_1^2Y_2 - \frac{1}{4}Y_2^3 - \frac{1}{2}Y_2Y_3^2},$$

$$y = \frac{-\frac{\sqrt{2}}{4}Y_1^2Y_3 - \frac{1}{2}Y_1^2Y_2 - \frac{1}{8}Y_2^3 - \frac{\sqrt{2}}{16}Y_2^2Y_3 + \frac{1}{2}Y_1Y_3^2 - \frac{\sqrt{2}}{8}Y_3^3}{-Y_1^2Y_2 - \frac{1}{4}Y_2^3 - \frac{1}{2}Y_2Y_3^2},$$

$$z = -\frac{\frac{\sqrt{2}}{2}Y_1^3 - Y_1^2Y_3 + \frac{\sqrt{2}}{4}Y_1Y_3^2 + \frac{\sqrt{2}}{8}Y_1Y_2^2 - \frac{1}{4}Y_2^2Y_3}{-Y_1^2Y_2 - \frac{1}{4}Y_2^3 - \frac{1}{2}Y_2Y_3^2}.$$

Usually, the rational parameterization of the cubic is very complicated so that a numerical computation for the algorithm to improve the computational efficiency is welcome.

5. Conclusion

We introduce some new concepts such as basic lines and divide the blending problems into two classes, Class I blending and Class II blending. We further develop the results in [5–8] and analyze the geometric condition for the Class I blending problem by using characteristic roots. We obtain the lowest degree of Class II blending according to the different position of the axes of the primary surfaces. And we derive the efficient parameterization method for cubic blending surfaces based on the results in [1].

All of our studies are based on the relevant position between some straight lines (such as basic lines and axes) and the primary quadratic surfaces. Such geometric method will give the designer more intuitive and easier way to the blending and parameterization of implicit algebraic surfaces than the typical pure algebraic way.

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